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Canonical Symmetries in the Functional Formalism

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Based on the phase-space generating functional of the Green function, the canonical Ward identities (CWI) under local, nonlocal, and global transformations in phase space for a system with a regular and singular Lagrangian have been derived. The relation of global canonical symmetries to conservation laws at the quantum level is presented. The advantage of this formulation is that one does not need to carry out the integration over canonical momenta in a phase-space path (functional) integral as in the traditional treatment in configuration space. In general, the connection between global canonical symmetries and conservation laws in classical theories is no longer preserved in quantum theories. Applications of our formulation to the non-Abelian Chern–Simons (CS) theory are given, and new forms for CS gauge-ghost field proper vertices and the quantal conserved angular momentum of this system are obtained; this angular momentum differs from the classical one in that one needs to take into account the contribution of angular momenta of ghost fields.

1. INTRODUCTION

The relation of global symmetries to conservation laws is usually referred to as the first Noether theorem.[1,2] This theorem says that to each symmetry of an action integral of the system there corresponds a conserved current.

The second Noether theorem refers to invariance of the action integral under an infinite continuous group (local symmetry), and this local invariance implies that there exist differential identities (Noether identities) for such a system. Noether identities correspond to Ward (or Ward–Takahashi) identities in quantum theory. Classical Noether theorems and Ward identities are usually formulated in terms of Lagrange's variables in configuration space.[3,4]

The invariance under the continuous group in terms of the canonical variables in classical theory has been developed in refs. 5–7. Ward identities

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relating the Green function in QED were obtained by Ward[8] and Takahashi.[9] In the non-Abelian theories their role is played by the so-called generalized Ward identities, first obtained by Slavnov[10] and Taylor.[11]

Ward identities and their generalization play an important role in modern quantum field theory. They are useful tools for the renormalization of field theory and for calculations in practical problems (for example, in QCD). Ward identities have been generalized to supersymmetry,[12] superstring theories,[13] and other problems.

All the derivations for Ward identities in the functional (path) integration method are usually performed by using a configuration-space generating functional, [14] which is valid for the case where the phase-space path integral can be simplified by carrying out explicit integration over canonical momenta; then the phase-space generating functional can be represented in the form of a path integral only over the coordinates (or field variables) of the expression containing a certain Lagrangian (or effective Lagrangian) in configuration space. In the case where the "mass" depends on coordinates[15,16] or on coordinates and canonical momenta,[17] one obtains an effective Lagrangian in the configuration-space path integral which shows singularities with a δ-function in both cases. Generally, for a constrained Hamiltonian system with complicated constraints[18] it is very difficult or even impossible to carry out the integration over canonical momenta in the phase-space path integral.

Phase-space path integrals are much more fundamental than configuration-space path integrals.[19] The latter provide a Hamiltonian quadratic in the canonical momenta, whereas the former apply to arbitrary Hamiltonians. The phase-space form of the path integral is a necessary precursor to the configuration-space form. The phase-space path integral formalism makes the symmetries of the system manifest in quantum theories.

The study of symmetry in the path-integration method has a more important sense. Preliminary discussions of the global and local canonical symmetry for a quantum system were given in refs. 20 and 21. In the quantum theories of the Yang–Mills field[22] and the conformal transformation of quantum fields in gauge theories[23,24] a nonlocal transformation was introduced and some applications given. In present paper, the nonlocal symmetries in phase space at the quantum level and other problems will be further studied.

The paper is organized as follows. In Section 2, based on the phase-space generating functional of the Green function, we derive the CWI under local and nonlocal transformations in phase space for a system with a regular or singular Lagrangian, respectively. This formulation differs from the traditional treatment in configuration space in that we do not need to carry out the integration over canonical momenta in the phase-space path integral. In Section 3 the CWI for global transformation is also deduced; the relations among Green functions can be obtained immediately. In Section 4 we find the quantum analogue of the first Noether theorem in the canonical formalism by subjecting the phase-space path integral to an infinitesimal global transformation of variables along the symmetry direction; the canonical Noether theorem at the quantum level has been established. In general, the connection between global symmetries and conservation laws in classical theories[25] is no longer preserved in quantum theories. In Section 5 the applications of the theory to the non-Abelian CS theory are given, a new form for CS gauge-ghost field proper vertices, and the conserved angular momentum at the quantum level of this system are obtained; this angular momentum differs from the classical one in that one needs to take into account the contribution of the angular momenta of ghost fields. The problem of fractional spin for the non-Abelian CS fields coupled to spinor fields needs further study.

2. CANONICAL WARD IDENTITIES FOR LOCAL AND NONLOCAL TRANSFORMATIONS

2.1. A System with a Regular Lagrangian

Let us first consider a physical field defined by the field variable $\varphi(x)$ and the motion of the field described by a regular Lagrangian $\mathcal{L}(\varphi, \varphi_{,\mu})$, $\varphi_{,\mu} = \partial_\mu \varphi = \partial\varphi/\partial x^\mu$, where $x = (t, \mathbf{x})$. The canonical Hamiltonian $H_c = \int d^3x\, \mathcal{H}_c = \int d^3x\, (\pi\dot{\varphi} - \mathcal{L})$ is a functional of independent canonical variables $\varphi(x)$ and $\pi(x)$, where $\pi(x) = \partial\mathcal{L}/\partial\dot{\varphi}$ is a canonical momentum conjugate to $\varphi(x)$. We adopt the path-integral quantization for the system; the phase-space generating functional of the Green function in the form of a path (functional) integral is[26]

$$Z[J, K] = \int \mathcal{D}\varphi\, \mathcal{D}\pi \exp i\left\{\left[I^p + \int d^4x\, (J\varphi + K\pi)\right]\right\} \tag{2.1}$$

where

$$I^p = \int d^4x\, \mathcal{L}^p = \int d^4x\, (\pi\dot{\varphi} - \mathcal{H}_c) \tag{2.2}$$

is the canonical action of the system, and $\mathcal{H}_c$ is the canonical Hamiltonian density. Here we have also introduced the exterior source K with respect to the canonical momentum π, which does not alter the calculation of the Green function G:

$$\begin{aligned} G(x_1, x_2, \ldots, x_n) &= \frac{1}{i^n} \frac{\delta^n Z[J, K]}{\delta J(x_1)\delta J(x_2) \cdots \delta J(x_n)}\bigg|_{J=K=0} \\ &= \langle 0 | T[\hat{\varphi}(x_1)\hat{\varphi}(x_2) \cdots \hat{\varphi}(x_n)] | 0\rangle \end{aligned} \tag{2.3}$$

Local gauge invariance is now a central concept in modern field theory, and nonlocal transformations in field theories also have been introduced.[22–24] First we consider the properties of the generating functional under general local and nonlocal transformations with the following form of infinitesimal transformation in extended phase space:

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + R^\mu_\sigma \varepsilon^\sigma(x) \\ \varphi(x') = \varphi(x) + \Delta\varphi(x) = \varphi(x) + A_\sigma \varepsilon^\sigma(x) + \displaystyle\int d^4x\, E(x, y) B_\sigma(y) \varepsilon^\sigma(y) \\ \pi'(x') = \pi(x) + \Delta\pi(x) = \pi(x) + U_\sigma \varepsilon^\sigma(x) + \displaystyle\int d^4y\, F(x, y) V_\sigma(y) \varepsilon^\sigma(y) \end{cases} \tag{2.4}$$

where $E(x, y)$ and $F(x, y)$ are some functions, and R^μ_σ, A_σ, B_σ, U_σ, and V_σ are linear differential operators,

$$R^\mu_\sigma = r^{\mu(l)}_\sigma \partial_{\mu(l)}, \qquad A_\sigma = a^{(m)}_\sigma \partial_{(m)}, \qquad B_\sigma = b^{(n)}_\sigma \partial_{(n)}, \qquad U_\sigma = u^{(p)}_\sigma \partial_{(p)}$$
$$V_\sigma = v^{(q)}_\sigma \partial_{(q)}, \qquad r^{\mu(l)}_\sigma = r_\sigma^{\overbrace{\mu\nu\cdots\lambda}^{l}}, \qquad a^{(m)}_\sigma = a_\sigma^{\overbrace{\mu\nu\cdots\rho}^{m}}, \qquad \text{etc.} \tag{2.5}$$

where $r^{\mu(l)}_\sigma$, $a^{(m)}_\sigma$, $b^{(n)}_\sigma$, $u^{(p)}_\sigma$, and $v^{(q)}_\sigma$ are functions of x, φ, and π, $\varepsilon^\sigma(x)$ ($\sigma = 1, 2, \ldots, s$) are arbitrary infinitesimal functions, and their values and derivatives up to the required order vanish on the boundary of the space-time domain. The variation of the canonical action (2.2) under the transformation (2.4) is given by[18]

$$\Delta I^p = \int d^4x \left\{ \frac{\delta I^p}{\delta\varphi} \delta\varphi + \frac{\delta I^p}{\delta\pi} \delta\pi + \partial_\mu[(\pi\dot{\varphi} - \mathcal{H}_c)\Delta x^\mu] + D(\pi\delta\varphi) \right\} \tag{2.6}$$

where $D = d/dt$, and

$$\frac{\delta I^p}{\delta\varphi} = -\dot{\pi} - \frac{\delta H_c}{\delta\varphi}, \qquad \frac{\delta I^p}{\delta\pi} = \dot{\varphi} - \frac{\delta H_c}{\delta\pi} \tag{2.7}$$

$$\delta\varphi = \Delta\varphi - \varphi_{,\mu}\Delta x^\mu, \qquad \delta\pi = \Delta\pi - \pi_{,\mu}\Delta x^\mu \tag{2.8}$$

The Jacobian of the transformation (2.4) is denoted by $J[\varphi, \pi, \varepsilon]$. The generating functional (2.1) is invariant under the transformation (2.4), which implies that $\delta Z/\delta\varepsilon^{\sigma}\big|_{\varepsilon^{\sigma}=0} = 0$. We substitute (2.4) and (2.6)–(2.8) into (2.1), integrate by parts corresponding terms, then functionally differentiate the results with respect to $\varepsilon^{\sigma}(x)$ and set $J = K = 0$, according to the boundary conditions of the functions $\varepsilon^{\sigma}(x)$; we obtain[27]

$$\int \mathcal{D}\varphi\, \mathcal{D}\pi \left\{ J_{\sigma}^{0} + \tilde{A}_{\sigma}(z)\left(\frac{\delta I^{p}}{\delta\varphi(z)}\right) + \tilde{U}_{\sigma}(z)\left(\frac{\delta I^{p}}{\delta\pi(z)}\right) \right.$$
$$- \tilde{R}_{\sigma}^{\mu}(z)\left[\varphi_{,\mu}(z)\frac{\delta I^{p}}{\delta\varphi(z)} + \pi_{,\mu}(z)\frac{\delta I^{p}}{\delta\pi(z)} \right]$$
$$+ \int d^{4}x\, \tilde{B}_{\sigma}(z)\left[E(x, z)\frac{\delta I^{p}}{\delta\varphi(x)} + D(\pi(x)E(x, z)) \right]$$
$$\left. + \int d^{4}x\, \tilde{V}_{\sigma}\left[F(x, z)\frac{\delta I^{p}}{\delta\pi(x)} \right] \right\} \exp(iI^{p}) = 0 \tag{2.9}$$

where

$$J_{\sigma}^{0} = -i\frac{\delta J[\varphi, \pi, \varepsilon]}{\delta\varepsilon^{\sigma}(z)}\bigg|_{\varepsilon^{\sigma}=0}$$

and $\tilde{A}_{\sigma}$, $\tilde{B}_{\sigma}$, $\tilde{R}_{\sigma}^{\mu}$, $\tilde{U}_{\sigma}$, and $\tilde{V}_{\sigma}$ are adjoint operators with respect to A_{σ}, B_{σ}, R_{σ}^{μ}, U_{σ}, and V_{σ}, respectively.[28] In deriving (2.9) we used the condition $J[\varphi, \pi, 0] = 1$. The Green function connected with (2.9) is given by

$$\langle 0 \big| T^{*} \left\{ J_{\sigma}^{0} + \tilde{A}_{\sigma}(z)\left(\frac{\delta I^{p}}{\delta\varphi(z)}\right) + \tilde{U}_{\sigma}(z)\left(\frac{\delta I^{p}}{\delta\pi(z)}\right) \right.$$
$$- \tilde{R}_{\sigma}^{\mu}(z)\left(\varphi_{,\mu}(z)\frac{\delta I^{p}}{\delta\varphi(z)} + \pi_{,\mu}(z)\frac{\delta I^{p}}{\delta\pi(z)} \right)$$
$$+ \int d^{4}x \left[\tilde{B}_{\sigma}(z)\left(E(x, z)\frac{\delta I^{p}}{\delta\varphi(x)} + D(\pi(x)E(x, z)) \right) \right.$$
$$\left.\left. + \tilde{V}_{\sigma}(x, z)\left(F(x, z)\frac{\delta I^{p}}{\delta\pi(x)} \right) \right] \right\} \big|0\rangle\big|_{\pi=\partial\mathcal{L}/\partial\dot{\varphi}} = 0 \tag{2.10}$$

where the symbol T^* stands for the covariantized T product,[14] and $|0\rangle$ is the vacuum state of the fields.

Substituting (2.4) and (2.6)–(2.8) into (2.1) and functionally differentiating the generating functional with respect to $\varepsilon^\sigma(x)$, we obtain

$$\Biggl\{J^0_\sigma + \tilde{A}_\sigma(z)\left(\frac{\delta I^p}{\delta\varphi(z)}\right) + \tilde{U}_\sigma(z)\left(\frac{\delta I^p}{\delta\pi(z)}\right) - \tilde{R}^\mu_\sigma(z)\Biggl[\varphi_{,\mu}(z)\left(\frac{\delta I^p}{\delta\varphi(z)} + J(z)\right) + \pi_{,\mu}(z)\left(\frac{\delta I^p}{\delta\pi(z)} + K(z)\right)\Biggr] + \int d^4x\Biggl[\tilde{B}_\sigma(z)\left(E(x,z)\left(\frac{\delta I^p}{\delta\varphi(x)} + J(x)\right) + D(\pi(x)E(x,z)) + \tilde{V}_\sigma(z)F(x,z)\left(\frac{\delta I^p}{\delta\pi(x)} + K(x)\right)\right)\Biggr]\Biggr\}\Bigg|_{\substack{\varphi\to(1/i)\delta/\delta J\\ \pi\to(1/i)\delta/\delta K}} Z[J,K] = 0 \tag{2.11}$$

Expression (2.11) is the canonical Ward identity (CWI) for local and nonlocal transformations. In the case of a local transformation ($E = F = 0$), the Jacobian of the corresponding transformation is independent of $\varepsilon^\sigma(x)$, which implies that $J^0_\sigma = 0$, and from (2.11) we have

$$\Biggl[\tilde{A}_\sigma\left(\frac{\delta I^p}{\delta\varphi}\right) - \tilde{R}^\mu_\sigma\left(\varphi_{,\mu}\frac{\delta I^p}{\delta\varphi}\right) + \tilde{U}_\sigma\left(\frac{\delta I^p}{\delta\pi}\right) - \tilde{R}^\mu_\sigma\left(\pi_{,\mu}\frac{\delta I^p}{\delta\pi}\right) + \tilde{A}_\sigma J - \tilde{R}^\mu_\sigma(\varphi_{,\mu}J) + \tilde{U}_\sigma K - \tilde{R}^\mu_\sigma(\pi_{,\mu}K)\Biggr]_{\substack{\varphi\to(1/i)\delta/\delta J\\ \pi\to(1/i)\delta/\delta K}} Z[J,K] = 0 \tag{2.12}$$

When we functionally differentiate expression (2.11) or (2.12) with respect to the exterior source J, we obtain another CWI. If we replace π by $\partial\mathcal{L}/\partial\dot{\varphi}$ in (2.11) or (2.12), these CWI can be expressed in terms of variables in configuration space, and we can obtain relations among the Green functions in which we do not need to carry out the integration over the canonical momenta in the phase-space generating functional (2.1).

2.2. A System with a Singular Lagrangian

Let us now consider a system with a singular Lagrangian $\mathcal{L}(\varphi^\alpha, \varphi^\alpha_{,\mu})$ whose Hess matrix $H_{\alpha\beta} = \partial^2\mathcal{L}/\partial\dot{\varphi}^\alpha\partial\dot{\varphi}^\beta$ is degenerate. Using the Legendre transformation, one can go over from the Lagrangian description to the Hamiltonian description, and the motion of the system is described by the canonial variables, subject to inherent phase-space constraints; this is called a constrained Hamiltonian system.[18,29–31] Let $\Lambda_k(\varphi^\alpha, \pi_\alpha) \approx 0$ ($k = 1, 2,$

$\ldots, K_0$) be first-class constraints, and $\theta_i(\varphi^\alpha, \pi_\alpha) \approx 0$ ($i = 1, 2, \ldots, I_0$) be second-class constraints. The gauge conditions connecting the first-class constraints are Ω_k ($k = 1, 2, \ldots, K_0$). According to the Faddeev–Senjanovic path-integral quantization method,[(32,33)] one can obtain the phase-space generating functional of a system with a singular Lagrangian as

$$\begin{aligned} Z[J, K, \eta^m, j, k, \bar{j}, \bar{k}] &= \int \mathscr{D}\varphi^\alpha\, \mathscr{D}\pi_\alpha\, \mathscr{D}\lambda_m\, \mathscr{D}C_a\, \mathscr{D}\pi^a\, \mathscr{D}\bar{C}_a\, \mathscr{D}\bar{\pi}^a \\ &\times \exp\left\{ i \left[I^p_{\text{eff}} + \int d^4x\, (J_\alpha \varphi^\alpha + K^\alpha \pi_\alpha + \eta^m \lambda_m \right.\right. \\ &\left.\left. + j^a C_a + k_a \pi^a + \bar{j}^a \bar{C}_a + \bar{k}_a \bar{\pi}^a \right] \right\} \end{aligned} \tag{2.13}$$

where

$$I^p_{\text{eff}} = \int d^4x\, \mathscr{L}^p_{\text{eff}} = \int d^4x\, (\mathscr{L}^p + \mathscr{L}_m + \mathscr{L}_{gh}) \tag{2.14}$$

$$\mathscr{L}^p = \pi_\alpha \dot{\varphi}^\alpha - \mathscr{H}_c \tag{2.15}$$

$$\mathscr{L}_m = \lambda_k \Lambda_k + \lambda_l \Omega_l + \lambda_i \theta_i \tag{2.16}$$

$$\mathscr{L}_{gh} = \int d^4y\, [\bar{C}_k(x)\{\Lambda_k(x), \Omega_l(y)\} C_l(y) + \tfrac{1}{2}\bar{C}_i(x)\{\theta_i(x), \theta_j(y)\} C_j(y)] \tag{2.17}$$

and $\lambda_m = (\lambda_k, \lambda_l, \lambda_i)$, and $C_a(x)$ and $\bar{C}_b(x)$ are Grassmann variables; $\pi^a(x)$ and $\bar{\pi}^b(x)$ are canonical momenta conjugate to $C_a(x)$ and $\bar{C}_b(x)$, respectively. J_α, K^α, η^m, j^a, k_a, $\bar{j}^a$, and $\bar{k}_a$ are exterior sources with respect to φ^α, π_α, λ_m, C_a, π^a, $\bar{C}_a$, and $\bar{\pi}^a$, respectively, and { , } denotes the Poisson bracket. For the sake of simplicity, let us denote $\varphi = (\varphi^\alpha, \lambda_m, C_a, \bar{C}_a)$, $\pi = (\pi_\alpha, \pi^a, \bar{\pi}^a)$, $J = (J_\alpha, \eta^m, j^a, \bar{j}^a)$, and $K = (K^\alpha, k_a, \bar{k}_a)$; thus, expression (2.17) can be written as

$$Z[J, K] = \int \mathscr{D}\varphi\, \mathscr{D}\pi\, \exp\left\{ i \left[I^p_{\text{eff}} + \int d^4x\, (J\varphi + K\pi) \right] \right\} \tag{2.18}$$

For a system with a singular Lagrangian, one can still proceed in the same way as with a regular Lagrangian to deduce the CWI under local and nonlocal transformations in phase space, but in this case one must use I^p_{eff} instead of I^p in the expressions (2.1), (2.2), (2.6), (2.7), and (2.9)–(2.12).

3. CANONICAL WARD IDENTITIES FOR GLOBAL TRANSFORMATION

Global symmetries such as Lorentz invariance, conformal symmetry, BRS and BRST invariance, supersymmetry, Siegel invariance,[34] etc., play an important role in field theories.[35] Here we further study the global canonical symmetry at the quantum level for a system with a singular Lagrangian.

Let us now consider an infinitesimal global transformation in extended phase space,

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + \varepsilon_\sigma \tau^{\mu\sigma}(x, \varphi, \pi) \\ \varphi'(x') = \varphi(x) + \Delta\varphi(x) = \varphi(x) + \varepsilon_\sigma \xi^\sigma(x, \varphi, \pi) \\ \pi'(x') = \pi(x) + \Delta\pi(x) = \pi(x) + \varepsilon_\sigma \eta^\sigma(x, \varphi, \pi) \end{cases} \tag{3.1}$$

where ε_σ ($\sigma = 1, 2, \ldots, r$) are infinitesimal arbitrary parameters, and $\tau^{\mu\sigma}$, ξ^σ, and η^σ are functions of x, $\varphi(x)$, and $\pi(x)$. The variation of the effective canonical action (2.14) is given by[18]

$$\Delta I^p_{\text{eff}} \int d^4x\, \varepsilon_\sigma \left\{ \left(-\dot{\pi} - \frac{\delta H_{\text{eff}}}{\delta \varphi} \right)(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma}) + \left(\dot{\varphi} - \frac{\delta H_{\text{eff}}}{\delta \pi} \right)(\eta^\sigma - \pi_{,\mu}\tau^{\mu\sigma}) \right.$$
$$\left. + \partial_\mu[(\pi\dot{\varphi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] + D[\pi(\xi^\sigma - \varphi_{,\mu}\tau_{\mu\sigma})] \right\} \tag{3.2}$$

where $H_{\text{eff}} = \int d^3x\, \mathcal{H}_{\text{eff}}$ is an effective Hamiltonian connected to the effective Lagrangian $L^p_{\text{eff}} = \int d^3x\, \mathcal{L}^p_{\text{eff}}$. It is supposed that the Jacobian of the transformation (3.1) is equal to unity, and the generating functional (2.18) is invariant under the transformation (3.1); thus we have

$$Z[J, K] = \int \mathcal{D}\varphi\, \mathcal{D}\pi \left\{ 1 + i\Delta I^p_{\text{eff}} + i\varepsilon_\sigma \int d^4x\, [J(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma}) \right.$$
$$\left. + K(\eta^\sigma - \pi_{,\mu}\tau^{\mu\sigma}) + \partial_\mu[(J\varphi + K\pi)\tau^{\mu\sigma}\Big]\right\}_{\substack{\varphi \to (1/i)\delta/\delta J \\ \pi \to (1/i)\delta/\delta K}} Z[J, K] = 0 \tag{3.3}$$

Therefore, the phase-space generating functional (2.18) satisfies

$$\int d^4x \left\{ \left(-\dot{\pi} - \frac{\delta H_{\text{eff}}}{\delta \varphi} \right)(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma}) + \left(\dot{\varphi} - \frac{\delta H_{\text{eff}}}{\delta \pi} \right)(\eta^\sigma - \pi_{,\mu}\tau^{\mu\sigma}) \right.$$
$$+ \partial_\mu[(\pi\dot{\varphi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] + D[\pi(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma})]$$

$$+ J(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma}) + K(\eta^\sigma - \pi_{,\mu}\tau^{\mu\sigma})$$

$$+ \partial_\mu[(J\varphi + K\pi)\tau^{\mu\sigma}]\Bigg\}_{\substack{\varphi\to(1/i)\delta/\delta J\\ \pi\to(1/i)\delta/\delta K}} Z[J, K] = 0 \tag{3.4}$$

If the effective canonical action (2.14) is invariant under the transformation (3.1), then the generating functional (2.18) satisfies

$$\int d^4x \left\{J\left(\xi^\sigma - \tau^{\mu\sigma}\partial_\mu \frac{\delta}{\delta J}\right) + K\left(\eta^\sigma - \tau^{\mu\sigma}\partial_\mu \frac{\delta}{\delta K}\right)\right.$$

$$\left. + \partial^\mu\left[\tau^{\mu\sigma}\left(J\frac{\delta}{\delta J} + K\frac{\delta}{\delta K}\right)\right]\right\}_{\substack{\varphi\to(1/i)\delta/\delta J\\ \pi\to(1/i)\delta/\delta K}} Z[J, K] = 0 \tag{3.5}$$

Expressions (3.4) and (3.5) are the CWI for the noninvariant and invariant systems under the global transformation in extended phase space, respectively. We functionally differentiate (3.4) or (3.5) with respect to exterior sources $J(x)$ many times and put exterior sources equal to zero, $J = K = 0$, to obtain relationships among the Green functions.

4. GLOBAL CANONICAL SYMMETRIES AND CONSERVATION LAWS AT THE QUANTUM LEVEL

The global canonical symmetries have been studied[5,7,25] in connection with the conservation laws in the canonical formalism at the classical level. Now we study the relation of the global canonical symmetries to the conservation laws at the quantum level for a system with a singular Lagrangian.

It is supposed that the variation of the effective canonical action (2.14) under the global transformation (3.1) is given by

$$\delta I^p_{\text{eff}} = \varepsilon_\sigma \int d^4x\, [\partial_\mu W^{\mu\sigma}(x, \varphi, \pi) + R^\sigma(x, \varphi, \pi)] \tag{4.1}$$

where $W^{\mu\sigma}$ and Q^σ are functions of x and canonical variables φ and π. Now we localize the transformation (3.1) and consider the following local transformation connected with the transformation (3.1):

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + \varepsilon_\sigma(x)\tau^{\mu\sigma}(x, \varphi, \pi) \\ \varphi'(x') = \varphi(x) + \Delta\varphi(x) = \varphi(x) + \varepsilon_\sigma(x)\xi^\sigma(x, \varphi, \pi) \\ \pi'(x') = \pi(x) + \Delta\pi(x) = \pi(x) + \varepsilon_\sigma(x)\eta^\sigma(x, \varphi, \pi) \end{cases} \tag{4.2}$$

where $\varepsilon_\sigma(x)$ ($\sigma = 1, 2, \ldots, r$) are infinitesimal arbitrary functions and their values and derivatives vanish on the boundary of the space-time domain. Under the transformation (4.2) the variation of the effective canonical action (2.14) is given by

$$\delta I^p_{\text{eff}} = \int d^4x\, \varepsilon_\sigma(x)\left\{\frac{\delta I^p_{\text{eff}}}{\delta\varphi}(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma}) + \frac{\delta I^p_{\text{eff}}}{\delta\pi}(\eta^\sigma - \pi_{,\mu}\tau^{\mu\sigma}) + \partial_\mu[(\pi\dot{\varphi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] + D[\pi(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma})]\right\} + \int d^4x\, \{(\pi\dot{\varphi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}\partial^\mu\varepsilon_\sigma(x) + \pi(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma})D\varepsilon_\sigma(x)\} \tag{4.3}$$

Since the variation of the effective canonical action under the global transformation (3.1) is given by (4.1), then in accord with the boundary conditions of $\varepsilon_\sigma(x)$, the expression (4.3) can be written as

$$\Delta I^p_{\text{eff}} = \int d^4x\, \varepsilon_\sigma(x)\{\partial_\mu[W^{\mu\sigma} - (\pi\dot{\varphi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] - D[\pi(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma})] + R^\sigma\} \tag{4.4}$$

Let us suppose tha the Jacobian of the transformation (4.2) is $J[\varphi, \pi, \varepsilon]$. The invariance of the generating functional (2.18) under the transformation (4.2) implies that

$$\left.\frac{\delta Z}{\delta t_\sigma(x)}\right|_{\varepsilon_\sigma(x)=0} = 0$$

Substituting (4.2) and (4.4) into (2.18) and functionally differentiating it with respect to $\varepsilon_\sigma(x)$, one obtains

$$\int \mathcal{D}\varphi\, \mathcal{D}\pi\, \{\partial_\mu[W^{\mu\sigma} - (\pi\dot{\varphi} - \mathcal{H}_{\text{eff}})\tau^{\mu\sigma}] - D[\pi(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma})] + R^\sigma + J_0^\sigma + M^\sigma\} \times \exp\left\{i\int d^4x\, (\mathcal{L}^p_{\text{eff}} + J\varphi + K\pi)\right\} = 0 \tag{4.5}$$

where

$$J_0^\sigma = -i\left.\frac{\delta J[\varphi, \pi, \varepsilon]}{\delta\varepsilon_\sigma(x)}\right|_{\varepsilon_\sigma(x)=0} \tag{4.6}$$

$$M^\sigma = J(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma}) + K(\eta^\sigma - \pi_{,\mu}\tau^{\mu\sigma}) \tag{4.7}$$

Functionally differentiating (4.5) with respect to $J(x)$ a total of n times, one obtains

$$\int \mathscr{D}\varphi\, \mathscr{D}\pi\, (\{\partial_\mu[W^{\mu\sigma} - (\pi\dot{\varphi} - \mathscr{H}_{\rm eff})\tau^{\mu\sigma}] - D[\pi(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma})] + R^\sigma$$

$$+ J_0^\sigma + M^\sigma\}\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n) - i\sum_j \varphi(x_1)\cdots\varphi(x_{j-1})\varphi(x_{j+1})\cdots\varphi(x_n)$$

$$\times N^\sigma\delta(x - x_j)\, \exp\left\{i\int d^4x\, (\mathscr{L}^p_{\rm eff} + J\varphi + K\pi)\right\} = 0 \tag{4.8}$$

where

$$N^\sigma = \xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma} \tag{4.9}$$

Let $J = K = 0$ in (4.8); one gets

$$[0\big|T^*\{\partial_\mu[W^{\mu\sigma} - (\pi\dot{\varphi} - \mathscr{H}_{\rm eff})\tau^{\mu\sigma}] - D[\pi(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma})]$$

$$+ R^\sigma + J_0^\sigma\}\varphi(x_1)\cdots\varphi(x_n)\big|0\rangle$$

$$= i\sum_j \langle 0\big|T^*[\varphi(x_1)\cdots\varphi(x_{j-1})\varphi(x_{j+1})\cdots\varphi(x_n)N^\sigma\big|0\rangle\delta(x - x_j) \tag{4.10}$$

where T^* stands for the covariantized T product.[3,14] Fixing t and letting

$$t_1, t_2, \ldots, t_m \to +\infty, \qquad t_{m+1}, t_{m+2}, \ldots, t_n \to -\infty$$

and using the reduction formula,[14] we can write equation (4.10) as

$$\langle out, m\big|\{\partial_\mu[W^{\mu\sigma} - (\pi\dot{\varphi} - \mathscr{H}_{\rm eff})\tau^{\mu\sigma}]$$

$$- D[\pi(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma})] + R^\sigma + J_0^\sigma\}\big|n - m, in\rangle = 0 \tag{4.11}$$

Since m and n are arbitrary, we have

$$\partial_\mu[W^{\mu\sigma} - (\pi\dot{\varphi} - \mathscr{H})_{\rm eff}\tau^{\mu\sigma} - D[\pi(\xi^\sigma - \varphi_{,\mu}\tau^{\mu\sigma})] + R^\sigma + J_0^\sigma = 0 \tag{4.12}$$

We take the integral of (4.12) on the 3-dimensional space; if we assume that the fields have a configuration which vanishes rapidly at spatial infinity, then using the Gauss theorem, we obtain

$$D\int d^3x\, [\pi(\xi^\sigma - \varphi_{,k}\tau^{k\sigma}) - \mathscr{H}_{\rm eff}\tau^{0\sigma} - W^{0\sigma}] = \int d^3x\, (R^\sigma + J_0^\alpha) \tag{4.13}$$

Consequently, we obtain the following theorem: If an effective Lagrangian $\mathscr{L}^p_{\rm eff}$ [see Eq. (2.14)] in phase space is invariant up to a 4-dimensional divergence term under the global transformation (3.1), i.e., $R^\sigma = 0$, in (4.1), and the Jacobian of the corresponding (4.2) is independent of $\varepsilon_\sigma(x)$, $J_0^\sigma = 0$, then there are conserved quantities at the quantum level for such a system with a singular Lagrangian:

$$Q^{\sigma} = \int d^3x\,[\pi(\xi^{\sigma} - \varphi_{,k}\tau^{k\sigma}) - \mathcal{H}_{\text{eff}}\tau^{0\sigma} - W^{0\sigma}] \tag{4.14}$$

These results hold true for anomaly-free theories. The conserved quantities (4.14) correspond to the classical conservation laws derived from the canonical Noether theorem.[36] For a system with a regular Lagrangian, there is no constraint in phase space for such a system. The generating functional for this system is given by (2.1). If the canonical action I^p is invariant under the global transformation in phase space and the Jacobian of the corresponding local transformation is equal to unity, then one can still proceed in the same way to obtain the quantal conserved quantities, but in this case one must use $\mathcal{H}_c$ instead of $\mathcal{H}_{\text{eff}}$ in expression (4.14). The connection between the symmetries and conservation laws at the quantum level differs from classical theory in that one must require that the Jacobian of the transformation be equal to unity. For a system with a singular Lagrangian, the canonical Noether theorem in classical field theory says that if the canonical action is invariant under a global transformation in phase space and the constraints (the equations of motion are determined by those constraints) are invariant under the substantial variation induced by such a global transformation, then there are conservation laws at the classical level. In the quantum theory, for the existence of the conserved quantities (4.14) one needs further to require that the whole constraints (including the gauge constraints) are invariant under the corresponding global transformation in phase space, i.e., the transformation must lie in the constrained hypersurface, so one can be sure that the effective canonical action is invariant under such a global transformation, and the Jacobian of the corresponding transformation is equal to unity. Therefore the relation of canonical symmetry to conservation laws at the quantum level is different from the classical case.

The advantage of this formulation is that one does not need to carry out explicitly the integration over the canonical momenta in the phase-space generating functional. Thus, this formulation can apply to the more general case.

5. NON-ABELIAN CHERN–SIMONS THEORY

Numerous recent work on (2+1)-dimensional gauge theories with Chern–Simons (CS) terms in the Lagrangian have revealed the occurrence of fractional spin and statistics.[37,38] These quantum theories are frequently used in condensed matter studies, such as the quantum Hall effect and high-T_c superconductivity.[39] theories with a non-Abelian CS term coupled to matter fields have also been studied.[40,41] Now we give some preliminary applications of the canonical symmetry at the quantum level to non-Abelian CS theory.

Proper Vertices

We start by considering the following singular Lagrangian density[42]:

$$\mathscr{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\theta}{4\pi}\,\varepsilon^{\mu\nu\rho}\left(\partial_\mu A^a_\nu A^a_\rho + \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\right) + i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi \tag{5.1}$$

describing the matter field ψ coupled to non-Abelian CS theories in (2+1) dimensions; $\psi = \psi^a T^a$, where the T^a are the generators of the gauge group. D_μ stands for the covariant derivative, and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \tag{5.2}$$

The gauge invariance of the non-Abelian CS term requires the quantization of the dimensionless θ, $\theta = n/4\pi$ ($n \in Z$).[43]

The canonical momenta connected with A^a_μ, ψ^a, and $\bar{\psi}^a$ are denoted by π^μ_a, $\bar{\pi}_a$, and π_a, respectively. The constraints for this model are[42]

$$\Lambda^a_1 = \pi^a_0 \approx 0 \tag{5.3}$$

$$\theta^a_1 = \bar{\pi}_a + i\bar{\psi}\gamma_0 \approx 0 \tag{5.4}$$

$$\theta^a_2 = \pi_a \approx 0 \tag{5.5}$$

$$\Lambda^a_2 = f^{abc}(\bar{\psi}^b \pi_c + \bar{\pi}_b \psi^c) + \partial_i \pi^i_a - f^{abc} A^b_i \pi^i_c + \frac{\theta}{4\pi}\,\varepsilon^{ij}\partial_i A^a_j \tag{5.6}$$

Λ^a_1 and Λ^a_2 are first-class constraints, θ^a_1 and θ^a_2 are second-class constraints.

According to the rule of path-integral quantization following the Faddeev–Senjanovic formalism,[32,33] for each first-class constraint, one must choose a gauge condition. The radiation gauge condition was chosen in ref. 42. However, this gauge condition is not consistent with the equations of motion of the system because of the existence of the matter field.[30,44] We choose other gauge conditions to study this problem. Consider the Coulomb gauge

$$\Omega^a_2 = \partial^i A^a_i \approx 0 \tag{5.7}$$

The consistent requirement of the gauge constraint $\Omega^a_2 \approx 0$, $\Omega^a_2 \approx 0$ implies another gauge constraint:

$$\Omega_1^a = \partial_i \pi_a^i + \nabla^2 A_0^a - f^{abc} A_i^b \partial^i A_0^c \approx 0 \tag{5.8}$$

It is easy to check that the factor $\{\theta_1^a, \theta_2^a\}$ is independent of field variables; thus, one can omit this factor from the generating functional. We have

$$\det\{\Lambda_k^a(x), \Omega_l^b(y)\} = \det M_c^{ab} \tag{5.9}$$

where

$$M_c^{ab} = (\delta^{ab}\nabla^2 - f^{abc} A_i^c \partial^i)\delta(x - y) \tag{5.10}$$

Since the theory is independent of the gauge, the factor $\delta(\partial^i A_i^a)\det M_c^{ab}$ can be replaced by $\delta(\partial^\mu A_\mu^a)\det M_L^{ab}$,[30,42] where

$$M_L^{ab} = (\delta^{ab}\partial^2 - f^{abc} A_\mu^c \partial^\mu)\delta(x - y) \tag{5.11}$$

Hence, the phase-space generating functional of the Green function for this model can be written as

$$\begin{aligned} Z[J, K, \bar{\eta}, K_1, \eta, \bar{K}_2, \bar{\zeta}, \zeta, \xi, X, Y] &= \int \mathcal{D} A_\mu^a \mathcal{D} \pi_a^\mu \mathcal{D} \psi^a \mathcal{D} \bar{\pi}_a \\ &\times \mathcal{D}\bar{\psi}^a \mathcal{D}\pi_a \mathcal{D}\bar{C}^a \mathcal{D} C^a \mathcal{D}\lambda_k^a \mathcal{D}\omega_k^l \mathcal{D}\nu_k^a \exp\left\{ i \int d^3x \,[I_{\text{eff}}^p + J_a^\mu A_\mu^a + K_\mu^a \pi_a^\mu \right. \\ &\left. + \bar{\eta}^a \psi^a + \bar{\pi}^a K_1 + \bar{\psi}^a \eta^a + \bar{K}_2^a \pi^a + \bar{\zeta}^a C^a + \bar{C}^a \zeta^a + \xi_k^a \lambda_k^a + X_k^l \omega_k^l + Y_k^a \nu_k^a] \right\} \end{aligned} \tag{5.12}$$

where

$$\mathcal{L}_{\text{eff}}^p = \mathcal{L}^p + \mathcal{L}_m + \mathcal{L}_{gh} \tag{5.13}$$

$$\mathcal{L}^p = \pi_a^\mu \dot{A}_\mu^a + \bar{\pi}^a \dot{\psi}^a + \dot{\bar{\psi}}^a \pi^a - \mathcal{H}_c \tag{5.14}$$

$$\mathcal{L}_m = \nu_k^a \theta_k^a + \lambda_k^a \Lambda_k^a + \omega_l^a \bar{\Omega}_l^a \qquad (\bar{\Omega}_1^a = \Omega_1^a, \quad \bar{\Omega}_2^a = \partial^\mu A_\mu^a) \tag{5.15}$$

$$\mathcal{L}_{gh} = \bar{C}^a M_L^{ab} C^b = -\partial^\mu \bar{C}^a D_{b\mu}^a C^b \tag{5.16}$$

It is easy to verify that the Lagrangian $\mathcal{L}^p + \mathcal{L}_{gh}$ is invariant under the following nonlocal transformation[22,45]:

$$\begin{cases} \psi'(x) = \psi(x) + i\varepsilon^{\sigma}(x)T_{\sigma}\psi(x), \quad \pi'(x) = \pi(x) - i\pi(x)\varepsilon^{\sigma}(x)T_{\sigma} \\ \bar{\psi}'(x) = \bar{\psi}(x) - \bar{\psi}(x)\varepsilon^{\sigma}(x)T_{\sigma}, \quad \bar{\pi}'(x) = \bar{\pi}(x) + i\varepsilon^{\sigma}(x)T^{\sigma}\bar{\pi}(x) \\ A_{\mu}^{a'}(x) = A_{\mu}^{a}(x) + D_{\sigma\mu}^{a}\varepsilon^{\sigma}(x), \quad \pi_{a}^{\mu'}(x) = \pi_{a}^{\mu}(x) + f_{\sigma}^{ac}\pi_{c}^{\mu}(x)\varepsilon^{\sigma}(x) \\ C^{a'}(x) = C^{a}(x) + i(T_{\sigma})^{ab}\varepsilon^{\sigma}(x)C^{b}(x) \\ \bar{C}^{a'}(x) = \bar{C}^{a}(x) - i\bar{C}^{b}(x)(T_{\sigma})^{ba}\varepsilon^{\sigma}(x) + \frac{i}{\Box}\partial_{\mu}[\bar{C}^{b}(x)(T_{\sigma})^{ba}\partial^{\mu}\varepsilon^{\sigma}(x)] \end{cases} \tag{5.17}$$

The last equation of (5.17) can be written as

$$\bar{C}^{a'}(x) = \bar{C}^{a}(x) - i\bar{C}^{b}(x)(T_{\sigma})^{ba}\varepsilon^{\sigma}(x) + \int d^3y\ \Delta_0(x, y)\partial_{\mu}[\bar{C}^{b}(y)(T_{\sigma})^{ba}\partial^{\mu}\varepsilon^{\sigma}(y)] \tag{5.18}$$

where

$$\Box\Delta_0(x, y) = i\delta^{(3)}(x - y) \tag{5.19}$$

The change of $\mathscr{L}_m$ up to a divergence term under the transformation (5.17) is given by

$$\delta\mathscr{L}_m = P_{\sigma}(\lambda, \omega, \nu, A, \pi, \ldots)\varepsilon^{\sigma}(x) \tag{5.20}$$

where P_{σ} are functions of multiplier fields, the CS field, the matter field, and ghost fields. The invariance of the generating functional (5.12) under the transformation (5.17) implies that

$$\begin{aligned} \Big\{ & J_{\sigma}^{0} + iP_{\sigma} + i\bar{\eta}^{a}T_{\sigma}\frac{\delta}{\delta\bar{\eta}_{a}} - iK_{1}^{a}T_{\sigma}\frac{\delta}{\delta K_{1}^{a}} - i\eta^{a}T_{\sigma}\frac{\delta}{\delta\eta^{a}} + i\bar{K}_{2}^{a}T_{\sigma}\frac{\delta}{\delta\bar{K}_{2}^{a}} \\ & - i\partial_{\mu}J_{\sigma}^{\mu} + f_{\sigma c}^{a}J_{a}^{\mu}\frac{\delta}{\delta J_{c}^{\mu}} + f_{\sigma c}^{a}K_{\mu}^{a}\frac{\delta}{\delta K_{\mu}^{c}} + i\bar{\zeta}^{b}(T_{\sigma})_{ab}\frac{\delta}{\delta\bar{\zeta}^{a}} \\ & - i\zeta^{a}(T_{\sigma})_{ba}\frac{\delta}{\delta\zeta^{b}} + \partial^{\mu}\left[\partial_{\mu}\left(\int d^3y\ \zeta^{a}\Delta_0(x, y)\right)(T_{\sigma})_{ba}\frac{\delta}{\delta\zeta^{b}}\right\} Z[J, K\ldots] = 0 \end{aligned} \tag{5.21}$$

where J_{σ}^{0} are independent of the field variables.[22,45] Let $Z[J, K, \ldots] = \exp\{iW[J, K, \ldots]\}$ and use the definition of the generating functional of proper vertices $\Gamma[A, \pi, \ldots]$ which is given by performing a functional Legendre transformation on $W[J, K, \ldots]$,

$$\Gamma[A_{\mu}^{a}, \pi_{a}^{\mu}, \ldots] = W[J_{a}^{\mu}, K_{\mu}^{a}, \ldots] - \int d^3x\ (J_{a}^{\mu}A_{\mu}^{a} + K_{\mu}^{a}\pi_{a}^{\mu} + \cdots) \tag{5.22}$$

$$\frac{\delta W}{\delta J_{a}^{\mu}(x)} = A_{\mu}^{a}(x), \qquad \frac{\delta\Gamma}{\delta A_{\mu}^{a}(x)} = -J_{a}^{\mu}(x) \tag{5.23a}$$

$$\frac{\delta W}{\delta K_\mu^a(x)} = \pi_a^\mu(x), \qquad \frac{\delta \Gamma}{\delta \pi_a^\mu(x)} = -K_\mu^a(x) \tag{5.23b}$$

$$\cdots \qquad\qquad \cdots$$

Thus, the expression (5.21) can be written as

$$\begin{aligned} &J_\sigma^0 + iP_\sigma - i\psi^a T_\sigma \frac{\delta\Gamma}{\delta\psi^a} + i\overline{\pi}_a T_a \frac{\delta\Gamma}{\delta\overline{\pi}_a} + i\overline{\psi}^a T_\sigma \frac{\delta\Gamma}{\delta\overline{\psi}^a} \\ &\quad -i\pi_a T_\sigma \frac{\delta\Gamma}{\delta\pi_a} + i\partial_\mu \frac{\delta\Gamma}{\delta A_\mu^\sigma} - f_{\sigma c}^a A_\mu^c \frac{\delta\Gamma}{\delta A_\mu^a} - f_{\sigma c}^a \pi_c^\mu \frac{\delta\Gamma}{\delta\pi_a^\mu} \\ &\quad -iC^a(T_\sigma)_{ab} \frac{\delta\Gamma}{\delta C^b} + i\overline{C}^a(T_\sigma)_{bc} \frac{\delta\Gamma}{\delta\overline{C}^b} \\ &\quad - i\partial^\mu \left[\partial_\mu \left(\int d^3y \frac{\delta\Gamma}{\delta\overline{C}^a} \Delta_0(x, y) \right) (T_\sigma)_{ba} \overline{C}^b \right] = 0 \end{aligned} \tag{5.24}$$

Functionally differentiating (5.24) with respect to $A^a(x_1)$ and $A_\lambda^b(x_2)$ and setting all fields (including the multiplier fields) equal to zero, one obtains

$$\partial_\mu \frac{\delta^3\Gamma[0]}{\delta A_\mu^\sigma(x)\delta A_\nu^a(x_1)\delta A_\lambda^b(x_2)} = if_{\rho\sigma}^a \delta(x - x_1) \frac{\delta^2\Gamma[0]}{\delta A_\nu^\rho(x)\delta A_\lambda^b(x_2)} \tag{5.25}$$

Functionally differentiating (5.24) with respect to $C^r(x_1)$ and $\overline{C}^s(x_2)$ and setting all fields (including the multiplier fields) equal to zero, one obtains

$$\begin{aligned} &(T_\sigma)_{rb}\delta(x - x_1) \frac{\delta^2\Gamma[0]}{\delta\overline{C}^s(x_1)\delta C^b(x)} - (T_\sigma)_{sb}\delta(x - x_2) \frac{\delta^2\Gamma[0]}{\delta\overline{C}^b(x)\delta C^r(x_1)} \\ &\quad + \partial_\mu \frac{\delta^3\Gamma[0]}{\delta\overline{C}^s(x_2)\delta C^r(x_1)\delta A_\mu^\sigma(x)} \\ &\quad + \partial^\mu \left[\partial_\mu \left(\int d^3y \frac{\delta^2\Gamma[0]}{\delta\overline{C}^a(x)\delta C^r(x_1)} \Delta_0(x, y)(T_\sigma)_{as}\delta(x - x_2) \right] = 0 \end{aligned} \tag{5.26}$$

Expressions (5.25) and (5.26) are Ward identities for CS gauge-host field vertices. This approach to obtaining the Ward identities for proper vertices has a significant advantage in that one does not need to carry out the integration over the canonical momenta in the phase-space path integral.

The effective Lagrangian (5.13) is invariant under spatial rotation, and the Jacobian of the transformation for field variables is equal to unity; from (4.14) we obtain the quantal conserved angular momentum

$$J_{lk} = \int d^2x \left\{ \pi_a^i \left(x_k \frac{\partial A_i^a}{\partial x_l} - x_l \frac{\partial A_i^a}{\partial x_k} \right) + \pi_a^i \left(\sum_{lk} \right)_{i\nu} A_a^\nu - i\bar{\psi}\gamma_0 S_{lk}\psi \right.$$
$$+ \bar{\pi} \left(x_k \frac{\partial\psi}{\partial x_l} - x_l \frac{\partial\psi}{\partial x_k} \right) + \bar{P}_a \left(x_k \frac{\partial C^a}{\partial x_l} - x_l \frac{\partial C^a}{\partial x_k} \right)$$
$$\left. + \left(x_k \frac{\partial \bar{C}^a}{\partial x_l} - x_l \frac{\partial \bar{C}^a}{\partial x_k} \right) P_a \right\} \tag{5.27}$$

where

$$\left(\sum_{\rho\sigma} \right)_{\mu\nu} = g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu} \tag{5.28}$$

$$S_{lk} = \tfrac{1}{4}[\gamma_l, \gamma_k] \tag{5.29}$$

and $\bar{P}_a$ and P_a are canonical momenta with respect to $\bar{C}^a$ and C^a, respectively.

From this result we see that the conserved angular momentum at the quantum level differs from the classical Noether one in that one needs to take into account the contribution of the angular momentum of ghost fields in the theories with a non-Abelian Chern–Simons term coupled to matter fields.

Similarly, we can proceed in the same way to study the BRS invariance in phase space; the Ward identity for the BRS transformation and the BRS-conserved quantity at the quantum level also can be deduced.

6. CONCLUSION AND DISCUSSION

In this paper we have studied the quantal canonical symmetry properties for a system with a singular Lagrangian. The path integrals provide a useful tool. The phase-space path integrals are more fundamental than the configuration-space path integrals. Based on the phase-space generating functional of the Green function obtained by using the Faddeev–Senjanovic path integral quantization method for a system with a singular Lagrangian, the canonical Ward identities under the local and nonlocal transformations in phase space for a system with a regular/singular Lagrangian are derived, respectively. The canonical Ward identities for the global transformation in phase space are also derived. The conservation laws at the quantum level in the canonical formalism for the global symmetry transformation are also deduced; in the general case these conservation laws differ from the classical Noether ones. A significant advantage of this formulation is that one does not need to carry out the integration over canonical momenta in the phase-space path integral as in the traditional treatment in configuration space.

The application of the theory to non-Abelian CS gauge fields coupled to a spinor field has been presented. The Ward identities for a nonlocal transformation have been derived. A new form of the Ward identity for the CS gauge-ghost proper vertices was obtained which differs from the Ward–Takahashi identity arising from the BRS invariance. The quantal conserved angular momentum arising from the invariance of spatial rotation was obtained, and differs from the result derived from the classical Noether theorem because one needs to take into account the contribution of the angular momentum of the ghost fields. Recent work[40] has studied the occurrence of fractional spin for non-Abelian CS theories in the classical case. Whether the fractional spin properties for non-Abelian CS theories are always valid at the quantum level needs further study.

For Abelian CS theories in the Coulomb gauge the ghost fields are absent in the path-integral quantization using the Faddeev–Senjanovic method for a constrained Hamiltonian system. From expression (4.14), one can obtain the quantal conserved angular momentum derived from the invariance of spatial rotation which coincides with the result derived from the classical Noether theorem.[37,38] Thus, the fractional spin and statistics in Abelian CS theories are preserved in quantum theories.

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